Precise Order of Magnitude in the Lp Local Limit Theorem

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Summary.

This paper is concerned with the upper and lower bounds of Lp metrics Δ_{np}, $1 \leqq p \leqq \infty$, constructed out of differences between density functions, for departure from normality for normed sums of independent and identically distributed random variables. It is shown that the Δ_{np} are asymptotically equivalent in the strong sense that, for $1 \leqq p, p' \leqq \infty$, $\Delta_{np'} / \Delta_{np}$ is universally bounded away from zero and infinity as $n \to \infty$.

1. Introduction and Results

There are many papers on the study of rates of convergence in the central limit theorem and the local limit theorem for normed sums of independent and identically distributed random variables. Hall [3] categorised rates of convergence results as belonging to one of the following two types:

(i) upper bounds on rates of convergence of the type derived by Lyapounov (1901), Berry (1941) and Esseen (1945); and

(ii) characterisations of the rate of convergence, like those obtained by Ibragimov (1966) and Heyde (1967).

He aimed at unifying these two disparate traditional approaches. Osipov [9, 10], Rozovskii [11], Hall [3, 4,] and Hall and Barbour [5] have studied the upper and lower bounds on rates of convergence. Heyde and Nakata [7] have shown equivalence of Lp metrics for convergence to normality over a broad range of cases.

In this paper, we shall obtain upper and lower bounds in the case of L_p metrics in the local limit theorem. Hall [3, 4] has already treated the case $p=\infty$. Moreover it is shown that the relation $\Delta_{np} \underset{\cap}{\cup} \delta_n$ (δ_n is clarified in theorem) holds, where $a_n \underset{\cap}{\cup} b_n$ for sequences of positive numbers $\{a_n\}$ $\{b_n\}$ means that $0 < \liminf_{n\to\infty} a_n/b_n \leqq \limsup_{n\to\infty} a_n/b_n < \infty$. This type of relation was first derived by Osipov [9].

Lex X_i $i=1, 2, \ldots$ be independent and identically distributed random variables with zero mean and variance unity and write $S_n = \sum_{i=1}^{n} X_i$,

$$\sigma_n^2 = E\{X^2 I(|X| \leqq n^{1/2})\} \text{ and } \nu_n = E\{XI(|X| \leqq n^{1/2})\}.$$

Suppose that for some n, S_n has a bounded density. Let $p_n(x)$ denote the density of $S_n/n^{1/2}$, $p_{nl}(x)$ denote the density of $(S_n - n\nu_n)/n^{1/2}\sigma_n$ and $\phi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$, $-\infty < x < \infty$.

We set, for any $A>0$ and $q(x) = p_n(x)$ or $p_{nl}(x)$

$$\Delta_{np}(A; q) = \left(\int_{-A}^{A} |p_n(x) - \phi(x)|^p (dx)\right)^{1/p}, \qquad 1 \leqq p < \infty,$$

$$\Delta_{n\infty}(\cdot\,; q) = \sup_x |p_n(x) - \phi(x)|, \qquad p=\infty,$$

Theorem 1 *If* $x^3 P(|X_1| > x) \to \infty$ *as* $x \to \infty$, *we have*

$$\Delta_{np}(\varepsilon; p_n) \underset{\cap}{\cup} \Delta_{np}(\infty; p_n) \underset{\cap}{\cup} \Delta_{np'}(\infty; p_n) \underset{\cap}{\cup} \delta_n$$

for each $\varepsilon > 0$, $1 \leqq p, p' \leqq \infty$.

where $\delta_n = EX^2 I(|X| > n^{1/2}) + n^{-1} EX^4 I(|X| \leqq n^{1/2})$

$$+ n^{-1/2} |EX^3 I(|X| \leqq n^{1/2})|.$$

Theorem 2 *If* $EX^4 = \infty$, *we have*

$$\Delta_{np}(\infty; p_{nl}) \underset{\cap}{\cup} \Delta_{np'}(\infty; p_{nl}) \underset{\cap}{\cup} \delta_{nl}$$

for $1 \leqq p,\ p' \leqq \infty$.

where $\delta_{n1} = nP(|X| > n^{1/2}) + n^{-1}E(X^4 I(|X| \leqq n^{1/2}))$

$$+ n^{-1/2}|EX^3 I(|X| \leqq n^{1/2})|.$$

Let X take only values of the form $a + Nb$ ($N = 0, \pm 1, \pm 3, \ldots$) where $b > 0$ is the maximal span of the lattice. Put

$$\Delta_{np}(A; \ell) = \sum_{N=-A}^{A} |b^{-1} n^{1/2} \sigma_n P(S_n = na + Nb) - \phi\{na + Nb - n\nu_n)/ n^{1/2}\sigma_n\}|^{p^{1/p}}$$

$$\Delta_{n\infty}(\cdot; \ell) = \sup_{-\infty < N < \infty} |b^{-1} n^{1/2} \sigma_n P(S_n = na + Nb) - \phi\{(na + Nb - n\nu_n)/ n^{1/2}\sigma_n\}|$$

Theorem 3 *If* $EX^4 = \infty$, *we have*

$$\Delta_{np}(\infty; \ell) \underset{\cap}{\cup} \Delta_{np'}(\infty; \ell) \underset{\cap}{\cup} \delta_{n1}$$

for $1 \leqq p,\ p' \leqq \infty$.

2. Some Lemmas

In order to prove these theorems, we need some lemmas. Lemmas 1, 3 and 4 are extensions of Theorem 5. 10 and Theorem 5. 1 of Hall [3] which treated the case $p = \infty$.

Lemma 1. *If* $EX^4 = \infty$, *then*

$$\Delta_{np}(\infty; p_n) \underset{\cap}{\cup} \delta_n, \quad (1 \leqq p \leqq \infty),$$

where $\delta_n = EX^2 I(|X| > n^{1/2}) + n^{-1} EX^4 I(|X| \leqq n^{1/2})$

$$+ n^{-1/2}|EX^3 I(|X| \leqq n^{1/2})|.$$

Lemma 2. *For each* $\varepsilon > 0$,

$$\liminf_{n \to \infty} \{\Delta_{np}(\varepsilon, p_n) + n^{-1/2}\} / \delta_n > 0, \quad (1 \leqq p \leqq \infty).$$

Lemma 3. *If* $EX^4=\infty$, *then*

$$\Delta_{np}\ (\infty;\ p_{nl.})\overset{\cup}{\cap}\delta_{nl},\quad (1\leqq p\leqq\infty).$$

where $\delta_{nl}=nP\,(|X|>n^{1/2})+n^{-1}E\,(X^4I\,(|X|\leqq n^{1/2}))$

$$+n^{-1/2}|EX^3I\,(|X|\leqq n^{1/2})|.$$

Lemma 4. *Let* $p_{nl}(x,\ c,\ d)$ *denote the density of* $(S_n-d)/cn^{1/2}$.

For each $\varepsilon>0,\ 1\leqq p\leqq\infty$,

$$\lim\,\inf_{n\to\infty}[\inf_{c>0,d}\Delta_{np}\,\{\infty;\ p_{nl}\,(x,\ c,\ d)\}+n^{-1}]/\delta_{nl}>0.$$

Lemma 5. *If* $EX^4=\infty$, *then*

$$\Delta_{np}\,(\infty;\ \ell)\overset{\cup}{\cap}\delta_{nl},\quad (1\leqq p\leqq\infty).$$

Lemma 6. *Suppose* X *takes only values of the form* $a+Nb\ (N=0,\pm 1,\pm 2,\cdots)$, *where* $b>0$ *is the maximal span of the lattice. Then*

$$\lim\,\inf_{n\to\infty}[\inf_{c>o,\ d}\{\sum_{N=-\infty}^{\infty}|b^{-1}cP\,(S_n=na+Nb)$$

$$-\phi\,\{(na-Nb-d)/c\}\,|^{p}\}^{1/p}+n^{-1}]/\delta_{nl}>0,$$

for $1\leqq p\leqq\infty$.

3. Proof of Theorems

We will establish Theorem 1 and Theorem 2. The proof of Theorem 3 closely resembles the proof of Theorems 1 and 2, and will not be given here.

Proof of Theorem 1. If $p\leqq p'$, by Lyapunov's inequality and Lemma 1, we have

$$\lim\,\inf_{n\to\infty}\delta_n^{\ 1}\Delta_{np}\,(\infty;\ p_n)\leqq\lim\,\sup_{n\to\infty}\delta_n^{\ 1}\Delta_{np}\,(\infty;\ p_n)$$

$$\leqq\lim\,\sup_{n\to\infty}\delta_n^{\ 1}\Delta_{np'}\,(\infty;\ p_n)\leqq C_2.$$

If $p' \leqq p$, we can use a similar argument. When $x^3 P(|X_1|>x) \to \infty$ as $x \to \infty$ we have from Lemma 2 that $n^{1/2}\Delta_{np}(\varepsilon; p_n) \to \infty$ as $n \to \infty$ since

$$n^{1/2}\delta_n \geqq n^{1/2} E X_1^2 I(|X_1|>n^{1/2}) \geqq n^{3/2} P(|X_1|>n^{1/2}) \to \infty$$

as $n \to \infty$. Therefore we have from Lemma 1 and Lemma 2, that there exist universal constants C_1 and C_2 such that

$$0<C_1 \leqq \liminf_{n\to\infty} \delta_n^{-1} \Delta_{np}(\varepsilon; p_n) \leqq \limsup_{n\to\infty} \delta_n^{-1} \Delta_{np}(\infty; p_n) \leqq C_2.$$

Proof of Theorem 2. In the proof of Lemma 4, we can choose c_n, d_n such that

$$\| p_{nl}(x; c_n, d_n) - \phi(x) \|_p \leqq 2 \{\inf_{c>o,\, d} \| p_{nl}(x; c, d) - \phi(x) \|_p\}$$

Noticing the definitions of p_{nl} and $p_{nl}(x; c, d)$, we may take $c_n = \sigma_n$, $d_n = n\nu_n$ such that

$$\liminf_{n\to\infty} [\Delta_{np}(\infty; p_{nl}) + n^{-1}] / \delta_{nl} > 0.$$

When $EX^4 = \infty$, we have from Lemma 4 that $n\Delta_{np}(\infty; p_{nl}) \to \infty$ as $n \to \infty$, since

$$n\delta_{nl} \geqq EX^4 I(|X|<n^{1/2}) \to \infty \tag{1}$$

as $n \to \infty$. Theorem 2 then follows from a minor modification of the proof of Theorem 1.

4. Proof of Lemmas.

Proof of Lemma 1. Hall [3] gives the following inequalities in Theorem 5. 3 (p209) and Theorem 5. 5 (p221).

$$\| p_n(x) - \phi(x) - L_n'(x) \|_p \leqq C_5 (\delta_n^2 + n^{-1}), \tag{2}$$

$$\| L_n'(x) \|_p \leqq C_6 \delta_n, \tag{3}$$

$$\lim\ \inf_{n\to\infty}\delta_n^{-1}\left(\int_{-\varepsilon}^{\varepsilon}|L_n'(x)|^p dx\right)^{1/p}>0, \tag{4}$$

where $L_n'(x)=nE\{\phi(x-X/n^{1/2})-\phi(x)\}-\frac{1}{2}\phi''(x)$.

From (3) and (4), we have $\| L_n'(x) \|_p \underset{\cap}{\cup} \delta_n$. But

$$\| p_n(x)-\phi(x) \|_p \leqq \| L_n'(x) \|_p + \| p_n(x)-\phi(x)-L_n'(x) \|_p,$$

$$\| p_n(x)-\phi(x) \|_p \geqq \| L_n'(x) \|_p - \| p_n(x)-\phi(x)-L_n'(x) \|_p,$$

and hence by using (2),

$$\| p_n(x)-\phi(x) \|_p \underset{\cap}{\cup} \delta_n,$$

since $EX^4=\infty$ ensures $n\delta_n \to \infty$ as $n \to \infty$.

Proof of Lemma 2. The proof of Lemma 2 has many similarities to that of Theorem 1. 3 in [3]. Let α denote the common characteristic function of the summands X_j. From the Fourier transformation, we have

$$\int_{-\infty}^{\infty} e^{itx}\{p_n(x)-\phi(x)\}dx=\alpha^n(t/n^{1/2})-e^{-t^2/2},$$

$$\int_{-\infty}^{\infty} e^{itx}\varepsilon^{-1}(1-|x/\varepsilon|)dx=2(1-\cos\varepsilon t)/(\varepsilon t)^2.$$

Applying Parseval's formula to these identities, we find that

$$\pi\int_{-\varepsilon}^{\varepsilon}\{p_n(x)-\phi(x)\}\varepsilon^{-1}(1-|x/\varepsilon|)dx.$$
$$=\int_{-\infty}^{\infty}\{\alpha^n(t/n^{1/2})-e^{-t^2/2}\}(1-\cos\varepsilon t)(\varepsilon t)^{-2}dt.$$

By using Holder's inequality, we have

$$\left|\int_{-\infty}^{\infty}\{\alpha^n(t/n^{1/2})-e^{-t^2/2}\}(1-\cos\varepsilon t)(\varepsilon t)^{-2}dt\right|\leqq C(\varepsilon)\Delta_{np}(\varepsilon; p_n);$$

while we know the following identity from Hall [3], page 15,

$$\alpha^n(t/n^{1/2})e^{t^2/2}=1+n\{\alpha(t/n^{1/2})-1+t^2/2n\}+r_n(t)$$

where $|r_n(t)|\leqq C(\delta_n^2+n^{-1})(t^4+t^8)e^{t^2/4}$

After a similar calculation to [3], page 17, we obtain

$$E\{X^2 I(|x|>n^{1/2})\}+n^{-1}E(X^4 I(|X|\leqq n^{1/2})$$

$$\leqq C\{\Delta_{np}(\varepsilon; p_n)+\delta_n^2+n^{-1/2}\}. \tag{5}$$

Next we consider the identity,

$$\int_{-\varepsilon}^{\varepsilon} e^{itx}(1-x^2/\varepsilon^2)\,dx=-4t^{-2}\varepsilon^{-1}\cos\varepsilon t+4t^{-3}\varepsilon^{-2}\sin\varepsilon t.$$

After a similar argument to [3] page 18, we have

$$n\left|\int_0^{n^{1/2}} E\{\sin(tX/n^{1/2})\}e^{-t^2/2}(4t^{-3}\varepsilon^{-2}\sin\varepsilon t-4t^{-2}\varepsilon^{-1}\cos\varepsilon t)\,dt\right|$$

$$\geqq n^{-1/2}|EX^3 I(|X|\leqq n^{1/2})|\{1+o(1)\}\int_0^{\infty}\frac{2}{3}e^{-t^2/2}(\sin\varepsilon t-t)\,dt$$

$$-C[EX^2 I(|X|>n^{1/2})+n^{-1}EX^4 I(|X|\leqq n^{1/2})].$$

Combining this inequality with (5) we may deduce that

$$\delta_n\leqq C\{\Delta_{np}(\varepsilon; p_n)+n^{-1/2}\},$$

which proves Lemma 2.

Proof of Lemma 3. In order to prove Lemma 3, it is sufficient to establish the following inequalities:

$$\sup_x(1+x^2)|p_{nl}(x)-\phi(x)-L'_{n11}(x)-L'_{n13}(x)|\leqq C(\delta_n\delta_{n1}^2+n^{-1}), \tag{6}$$

$$\begin{aligned}&\sup_x(1+x^2)|L'_{n11}(x)|\leqq C\delta_{nl},\\&\sup_x(1+x^2)|L'_{n13}(x)|\leqq C\,\delta_n\delta_{nl},\end{aligned} \tag{7}$$

$$\liminf_{n\to\infty}\delta_{nl}^{-1}\left(\int_{-\infty}^{\infty}|L'_{n11}(x)|^p dx\right)^{1/p}>0, \tag{8}$$

where $L'_{n11}(x)=nE\{\phi(x-X/n^{1/2}\sigma_n)-\phi(x)\}$

$$+n^{1/2}\nu_n\phi'(x)/\sigma_n-\frac{1}{2}\phi''(x),$$

$$L'_{n12}(x)=nE\{\phi(x-X/n^{1/2}\sigma_n)-\phi(2^{1/2}x)\}$$

$$+(n^{1/2}\nu_n/\sigma_n)\, 2^{1/2}\phi'(2^{1/2}x)-\phi''(2^{1/2}x),$$

$$L'_{n13}(x)=L'^{*2}_{n12}/2, \text{ (the second convolution of } L'_{n12}).$$

these quantities being defined in [3], and C being a universal constant.

Actually, from (6) and (7), we have

$$\| p_{nl}(x)-\phi(x)-L'_{n11}(x)-\hat{L}_{n13}(x)\|_p \leqq C(\delta_n\delta_{n1}^2+n^{-1}),$$

$$\| L'_{n11}(x)\|_p \leqq C\,\delta_{nl},\quad \| L'_{n13}(x)\|_p \leqq C\,\delta_n\delta_{nl}.$$

While

$$\| L'_{n11}(x)-L'_{n13}\|_p \leqq \| L'_{n11}(x)\|_p+\| L'_{n13}(x)\|_p$$
$$\leqq C\,\delta_{nl}(1+\delta_n), \tag{9}$$

$$\| L'_{n11}(x)-L'_{n13}\|_p \geqq \| L'_{n11}(x)\|_p-\| L'_{n13}(x)\|_p$$
$$\geqq C\,\delta_{nl}(1-\delta_n), \tag{10}$$

since we have $\| L'_{n11}(x)\|_p \geqq C\,\delta_{nl}$ from (8).

Since

$$\| p_{nl}(x)-\phi(x)\|_p \leqq \| L'_{n11}(x)-L'_{n13}\|_p$$
$$+\| p_{nl}(x)-\phi(x)-L'_{n11}(x)-L'_{n13}(x)\|_p,$$

$$\| p_{nl}(x)-\phi(x)\|_p \geqq \| L'_{n11}(x)-L'_{n13}\|_p$$
$$-\| p_{nl}(x)-\phi(x)-L'_{n11}(x)-L'_{n13}(x)\|_p,$$

in view of (9) and (10), we have

$$\| p_{nl}(x)-\phi(x)\|_p \leqq C(\delta_{nl}+n^{-1}),$$

$$\| p_{nl}(x)-\phi(x)\|_p \geqq C(\delta_{nl}-n^{-1}).$$

and hence

$$\| p_{nl}(x)-\phi(x)\|_{p} \asymp \delta_{nl},$$

since $EX^4=\infty$ ensures that $n\delta_{nl}\to\infty$ as $n\to\infty$.

Now (6) and the latter half of (7) were proved in Hall [3], p210 and p118. We will show the first half of (7) and (8). From the definition of $L'_{n11}(x)$, we have

$$(1+x^2)|L'_{n11}(x)|$$

$$\leqq(1+x^2)|nE[\{\phi(x-X/n^{1/2}\sigma_n)-\phi(x)+X\phi'(x)/n^{1/2}\sigma_n$$

$$-\frac{1}{2}(X/n^{1/2}\sigma_n)^2\phi''(x)\}I(|X|\leqq n^{1/2})]$$

$$+nE\{\phi(x-X/n^{1/2}\sigma_n)-\phi(x)\}I(|X|>n^{1/2})|,$$

$$\leqq(1+x^2)(1/6\sigma_n^3)|\phi^{(3)}(x)\| n^{-1/2}EX^3I(|X|<n^{1/2})|$$

$$+(1/12\sigma_n^4)\sup_{x-1/\sigma_n\leqq y\leqq x+1/\sigma_n}|(1+y^2+\sigma_n^{-1})\phi^{(4)}(y)\| n^{-1}EX^4I(|X|\leqq n^{1/2})|$$

$$+(1+x^2)\, n\, E\phi(x-X/n^{1/2}\sigma_n)I(|X|>n^{1/2})$$

$$+(1+x^2)|\phi(x)|nP(|X|>n^{1/2})$$

$$\leqq C\delta_{nl}.$$

(8) folows from a minor modification of the proof of Theorem 3. 6 of [3] wherein it is necessary to replace (3.84), (3.85) and b_z by

$$\int_{-\infty}^{\infty}\{\phi(x-u)-\phi(x)+u\phi'(x)-\tfrac{1}{2}u^2\phi''(x)\}e^{itu}dx$$

$$=\{e^{itu}-1+itu-\tfrac{1}{2}(itu)^2\}e^{-t^2/2}.$$

$$\int_{-\infty}^{\infty}\{\phi(x-\nu)-\phi(x)\}e^{itx}dx=\{e^{it\nu}-1\}e^{-t^2/2}$$

and

$$b_z = \begin{cases} (z-t)\, e^{t^2/2} & \text{if } 0 \leqq t \leqq z \\ 0 & \text{otherwise} \end{cases}$$

respectively.

Proof of Lemma 4. This proof has much similarity to the proof of Theorem 3.1 of [3]. We can choose c_n, d_n such that

$$\| p_{nl}(x;\, c_n,\, d_n) - \phi(x) \|_p \leqq 2 \{ \inf_{c>o,\, d} \| p_{nl}(x;\, c,\, d) - \phi(x) \|_p \}.$$

Since $\| p_{nl}(x;\, 1,\, 0) - \phi(x) \|_p \to 0$ as $n \to \infty$ then it is necessarily true that $c_n \to 1$ and $d_n \to 0$ as $n \to \infty$. If we define

$$\beta_n(t) = \int_{-\infty}^{\infty} e^{itx} p_{nl}(x;\, c_n,\, d_n)\, dx,$$

$$b_z(t) = \begin{cases} (z-t)\, t^2 e^{t^2/2} & \text{if } 0 \leqq t \leqq z, \\ 0 & \text{otherwise,} \end{cases}$$

we need to replace (3. 5) of [3] by

$$\beta_n(t) - e^{-t^2/2} = \int_{-\infty}^{\infty} e^{itx} \{ p_{nl}(x;\, c_n,\, d_n) - \phi(x) \}\, dx.$$

Moreover, we replace an inequality, 4 line in page 97 of [3] by

$$\int_{-\infty}^{\infty} |\beta_n(t) - e^{-t^2/2}|^2 dt \leqq 2\pi\, K_1^{\,1-1/p} \{ \int_{-\infty}^{\infty} |p_{nl}(x;\, c_n,\, d_n) - \phi(x)|^p dx \}^{1/p}$$
$$\cdot \{ \sup_x |p_{nl}(x;\, c_n,\, d_n) - \phi(x)| \}^{1/p},$$

where $K_1 = \int_{-\infty}^{\infty} |p_{nl}(x;\, c_n,\, d_n) - \phi(x)|\, dx$. After a similar argument to that of Hall, we obtain Lemma 4.

Proof of Lemma 5. Set

$$x_{nN} = (na + Nb - n\nu_n) / n^{1/2} \sigma_n$$

and

$$p_{nN} = P\{ (S_n - n\nu_n) / n^{1/2} \sigma_n = x_{nN} \}$$

for $-\infty<N<\infty$. With this notation, in order to prove Lemma 5, it is sufficient to prove inequalities like (6), (7), (8). However, we know (5. 2) of [3] and we can show

$$\liminf_{n\to\infty}\delta_{n1}^{-1}\left(\sum_{N=-\infty}^{\infty}|L'_{n11}(x_{nN})|^{p}\right)^{1/p}>0$$

in a similar fashion to the proof of Theorem 5.8 of [3].

Proof of Lemma 6. This proof follows from a minor modification of the proof of Theorem 5.2 of [3].

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